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ASPECTS OF VARIATIONAL ARGUMENTS IN THE THEORY OF ELASTICITY: FACT AND FOLKLORE

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Abstract—An account of the use of the calculus of variations in elastostatics is given. Variations of both the independent and dependent variables are used to obtain the equations of balance and natural boundary conditions. The meaning and use of the necessary transversality conditions for movable internal or external boundaries are examined. Satisfaction of the transversality conditions is shown to provide immediate access to the forces that act on such boundaries through a virtual work type argument. Misconceptions concerning the general form of Lagrangian densities for which the field equations are identically satisfied are corrected. The complete characterization of such Lagrangian densities is shown to lead to methods for realization of a wide class of traction boundary conditions as natural boundary conditions and to a significantly richer class of invariance transformations and associated conservation laws.

A perusal of the literature in elastostatics and elastodynamics indicates the presence of certain fundamental gaps when variational arguments are employed. This does not mean that the results obtained by various variational arguments are wrong; rather, they are just incomplete. Unfortunately, the incomplete results have often evolved over the years into what may be termed folklore, for they are mistakenly used in conjunction with that most abused work "all" to imply that the results are exhaustive. This is particularly true with respect to conservation laws and with respect to the class of Lagrangian functions whose Euler–Lagrange equations are identically satisfied. There is also an aspect of the calculus of variations, namely the need to satisfy transversality conditions in order to render the action functional stationary in value in the presence of mobile internal or external boundaries, that seems to have been completely overlooked. It is hoped that the exposition given here will aid in clarifying these points and be of use to those who deal with variational arguments in elasticity theory.

1. PRELIMINARIES AND NOTATION

The first thing to do is to agree upon a convenient system of notation. Since either Eulerian (x^i) or Lagrangian (X^i) coordinates can be employed in the description of material bodies, it seems preferable at the outset to choose a notation that is not specific to either choice and hence can be used with both. The independent coordinate variables are denoted by $(\tau^i) = (\tau^1, \tau^2, \ldots, \tau^n)$ and are considered as Cartesian coordinates of an n-dimensional Euclidean space, E_n. The value of n for most of this paper will be 3, since attention will be confined to elastostatics. The results are also applicable to the case of elastodynamics (n equal to 4), the only change being one of terminology.

The elastic body, $\mathcal{B}$, under consideration is assumed to occupy an n-dimensional closed and connected point set, B, of E_n in some "standard" configuration, $\mathcal{B}_0$, with a nonzero volume measure, $\int_B \mathrm{d}V$. Here, $\mathrm{d}V$ stands for the differential volume element $\mathrm{d}\tau^1\,\mathrm{d}\tau^2 \ldots \mathrm{d}\tau^n$. We also assume that B has a smooth boundary, ∂B (i.e. an outward oriented unit normal vector field with components $\{n_i\}$ is defined at every point ∂B). The total flux of a vector field with components $\{T^i\}$ out of the body is thus given by $\int_{\partial B} T^i n_i \,\mathrm{d}S$, where $\{n_i\,\mathrm{d}S\}$ denote the components of the outward oriented differential boundary elements of B. The summation convention is explicitly assumed with respect to lower case Latin indices.

The geometric state of a deformed elastic body is described by a collection of fields that are functions of the independent variables. Since it is not necessary to be too specific at this point, let us suppose that there are N such fields which we denote by $\{\phi^\alpha(\tau^m)\} = \{\phi^1(\tau^m), \phi^2(\tau^m), \ldots, \phi^N(\tau^m)\}$. Again, in practice, N will be 3, but just what the names that are to be attached to the ϕ^α's will vary, depending upon whether the description is an Eulerian or a Lagrangian one and whether displacements are to be introduced in such descriptions.

The energetic state of a deformed elastic body requires the values of the derivatives of the state variables as well as the state variables themselves. A convenient notation for these is

$$y_i^\alpha(\tau^m) = \frac{\partial \phi^\alpha(\tau^m)}{\partial \tau^i} \tag{1.1}$$

since it eliminates the need to write the cumbersome expressions $\partial\phi^\alpha/\partial\tau^i$ for each and every occurrence of such arguments. Alghough this notation is not standard, it has certain intrinsic advantages that will become evident shortly.

If we have a function $f(\tau^m, \phi^\alpha(\tau^m), y_i^\alpha(\tau^m))$, it is essential that a careful distinction be made between the total partial derivative with respect to τ^k and the explicit partial derivative with respect to τ^k. We shall use D_k to denote the total partial derivative and ∂_k to denote the explicit partial derivative with respect to τ^k (i.e. the partial derivative with respect to τ^k with ϕ^α and y_i^α held constant). These are connected by the relations

$$D_k f(\tau, \phi, y) = \partial_k f + y_k^\alpha \frac{\partial f}{\partial \phi^\alpha} + y_{ik}^\alpha \frac{\partial f}{\partial y_i^\alpha}, \tag{1.2}$$

where the summation is assumed with respect to both Latin and Greek indices over their respective ranges and we have introduced the further notational convenience

$$y_{ik}^\alpha(\tau^m) = \frac{\partial^2 \phi^\alpha(\tau^m)}{\partial \tau^k \partial \tau^i} = y_{ki}^\alpha(\tau^m). \tag{1.3}$$

2. LAGRANGIANS AND VARIATIONS

We explicitly assume throughout this paper that the elastic body $\mathscr{B}$ may be assigned a Lagrangian density function

$$L(\tau^m, \phi^\alpha(\tau^m), y_i^\alpha(\tau^m)).$$

This serves to define the action of $\mathscr{B}$ through the relation

$$A[\phi^\alpha] = \int_B L(\tau^m, \phi^\alpha(\tau^m), y_i^\alpha(\tau^m))\, \mathrm{d}V. \tag{2.1}$$

Since the Lagrangian density is, by definition, the kinetic energy density *minus* the potential energy density, a homogeneous elastic body with strain energy density $W(y_i^\alpha)$ and conservative body forces that derive from a potential $\mathscr{V}(\phi^\alpha)$ would give

$$L = -W - \mathscr{V}, \tag{2.2}$$

in which case

$$\partial L/\partial \phi^\alpha = -\partial \mathscr{V}/\partial \phi^\alpha, \quad \partial L/\partial y_i^\alpha = -\partial W/\partial y_i^\alpha.$$

It is therefore useful to introduce

$$f_\alpha = \partial L/\partial \phi^\alpha \tag{2.3}$$

($f_\alpha = -\partial \mathscr{V}/\partial \phi^\alpha$) as the components of generalized body forces and

$$\sigma_\alpha{}^i = -\partial L/\partial y_i^\alpha \tag{2.4}$$

($\sigma_\alpha{}^i = \partial W/\partial y_i^\alpha$) as the components of generalized stress. For the Lagrangian description in which τ^i are the reference configuration variables, X^i and ϕ^α are the current configuration variables, $x^i(X^j)$, we have $y_j^i = \partial x^i/\partial X^j$ and $\sigma_i^j = -\partial L/\partial y_j^i = \partial W/\partial(\partial x^i/\partial X^j)$ which are just the components of the Piola–Kirchhoff stress.

The primitive notion of the calculus of variations is that we compare the value $A[\phi^\alpha]$ of the action at the state $\{\phi\}$ with the value $A[\bar{\phi}^\alpha]$ of the action at a "neighboring" state, it being assumed that the independent variables are not changed. This, unfortunately, is not adequate if one is to consider problems with variable boundaries or wishes to derive conservation laws that will be satisfied whenever the "equations of motion" (Euler–Lagrange equations) are satisfied. It is therefore simplest to start on the right track from the very beginning by considering "variations" that obtain as a consequence of changes in both the independent and the dependent variables. To this end, we consider a 1-parameter family of transformations

$$\bar{\tau}^i(\lambda) = P^i(\tau^m, \phi^\beta; \lambda), \bar{\phi}^\alpha(\lambda) = P^\alpha(\tau^m, \phi^\beta; \lambda), \tag{2.5}$$

where the functions $\{P^i, P^\alpha\}$ are very smooth functions of the parameter λ for all $\{\tau^m, \phi^\beta\}$ and satisfy the conditions

$$P^i(\tau^m, \phi^\beta; 0) = \tau^i, P^\alpha(\tau^m, \phi^\beta; 0) = \phi^\alpha. \tag{2.6}$$

In fact, we may, without loss of generality, assume that the $\{\bar{\tau}^i, \bar{\phi}^\alpha\}$ are solutions of the autonomous system of ordinary differential equations

$$\frac{\mathrm{d}\bar{\tau}^i}{\mathrm{d}\lambda} = v^i(\bar{\tau}^m, \bar{\phi}^\beta), \frac{\mathrm{d}\bar{\phi}^\alpha}{\mathrm{d}\lambda} = v^\alpha(\bar{\tau}^m, \bar{\phi}^\beta) \tag{2.7}$$

subject to the initial data

$$\bar{\tau}^i(0) = \tau^i, \bar{\phi}^\alpha(0) = \phi^\alpha, \tag{2.8}$$

where the functions (v^i, v^α) are very smooth (infinitely differentiable) functions of their arguments. Under these conditions, we have

$$\begin{aligned} \bar{\tau}^i &= \tau^i + \lambda v^i(\tau^m, \phi^\beta) + 0(\lambda^2), \\ \bar{\phi}^\alpha &= \phi^\alpha + \lambda v^\alpha(\tau^m, \phi^\beta) + 0(\lambda^2), \end{aligned} \tag{2.9}$$

and hence the variations (linear departures) are given by

$$\delta\tau^i = v^i(\tau^m, \phi^\beta), \delta\phi^\alpha = v^\alpha(\tau^m, \phi^\beta) \tag{2.10}$$

(i.e. $\bar{\tau}^i = \tau^i + \lambda\delta\tau^i + 0(\lambda^2)$, $\bar{\phi}^\alpha = \phi^\alpha + \lambda\delta\phi^\alpha + 0(\lambda^2)$). It is then a simple matter to compute $\bar{y}_i^\alpha = \partial\bar{\phi}^\alpha/\partial\bar{\tau}^i$ and then express the results in terms of the functions $\{v^i, v^\alpha\}$ or $\{\delta\tau^i, \delta\phi^\alpha\}$:

$$\begin{aligned} \bar{y}_i^\alpha &= y_i^\alpha + \lambda\left(\partial_i + y_i^\gamma \frac{\partial}{\partial\phi^\gamma}\right)(v^\alpha - y_k^\alpha v^k) + 0(\lambda^2) \\ &= y_i^\alpha + \lambda\left(\partial_i + y_i^\gamma \frac{\partial}{\partial\phi^\gamma}\right)(\delta\phi^\alpha - y_k^\alpha\delta\tau^k) + 0(\lambda^2). \end{aligned} \tag{2.11}$$

Here and throughout this paper, we have adopted the convention that ∂_i and $\partial/\partial\phi^\alpha$ give zero when applied to the variables y_i^α (i.e. $(\tau^k, \phi^\alpha, y_i^\alpha)$ are to be taken as independent of each other in all differentiation processes). Thus (2.11) is an abbreviation for

$$\bar{y}_i^\alpha = y_i^\alpha + \lambda\left\{\partial_i v^\alpha - y_k^\alpha\partial_i v^k + y_i^\gamma\frac{\partial v^\alpha}{\partial\phi^\gamma} - y_i^\gamma y_k^\alpha\frac{\partial v^k}{\partial\phi^\gamma}\right\} + 0(\lambda^2).$$

The variations that are induced in the y's by the variations (2.10) in the τ's and in the ϕ's are thus given by

$$\delta y_i^\alpha = \left(\partial_i + y_i^\gamma \frac{\partial}{\partial \phi^\gamma}\right)(\delta\phi^\alpha - y_k^\alpha \delta\tau^k). \tag{2.12}$$

If we choose the functions $\{v^i, v^\alpha\}$ by

$$v^i = 0, v^\alpha = h^\alpha(\tau^m),$$

then (2.10) and (2.12) give

$$\delta\tau^i = 0, \delta\phi^\alpha = h^\alpha(\tau^m), \delta y_i^\alpha = \partial_i h^\alpha(\tau^m),$$

in which case $\delta y_i^\alpha = \partial_i \delta\phi^\alpha$ (variation of the derivative is equal to the derivative of the variation). It is, however, essential to note that $\delta y_i^\alpha \neq \partial_i \delta\phi^\alpha$ in the general case, as is immediately evident from (2.12). Further, if we require that the independent variables remain unvaried (2.12) shows that

$$\delta y_i^\alpha = \partial_i \delta\phi^\alpha + y_i^\gamma \frac{\partial \delta\phi^\alpha}{\partial \phi^\gamma} = D_i \delta\phi^\alpha,$$

as it should be, rather than $\partial_i \delta\phi^\alpha$ which is what might be written if one is not careful to distinguish between the explicit partial derivative, ∂_i, and the total partial derivative, D_i.

Let $\bar{B}$ denote the image of the region B of E_n under the mapping defined by (2.5) and let

$$\mathrm{d}\bar{V} = \mathrm{d}\bar{\tau}^1 \,\mathrm{d}\bar{\tau}^2 \ldots \mathrm{d}\bar{\tau}^n$$

denote the differential element of volume with reference to the new coordinate cover $(\bar{\tau}^i)$. We then have the new action functional

$$A[\bar{\phi}] = \int_{\bar{B}} L(\bar{\tau}^m, \bar{\phi}^\alpha, \bar{y}_i^\alpha)\,\mathrm{d}\bar{V} \tag{2.13}$$

that is to be compared with $A[\phi^\alpha]$. When all quantities that occur in (2.13) are expanded in ascending powers of the parameter λ and note is taken of the fact that

$$\mathrm{d}\bar{V} = (1 + \lambda D_k \delta\tau^k + 0(\lambda^2))\,\mathrm{d}V,$$

we obtain

$$A[\bar{\phi}^\alpha] = A[\phi^\alpha] + \lambda \int_B \left\{ L D_k \delta\tau^k + (\partial_k L)\delta\tau^k + \frac{\partial L}{\partial \phi^\alpha}\delta\phi^\alpha + \frac{\partial L}{\partial y_i^\alpha}\delta y_i^\alpha \right\} \mathrm{d}V + 0(\lambda^2). \tag{2.14}$$

The definition of the variational derivative of A,

$$\delta A = \lim_{\lambda \to 0}\left(\frac{A[\bar{\phi}^\alpha] - A[\phi^\alpha]}{\lambda}\right),$$

then gives

$$\delta A = \int_B \left\{ L D_k \delta\tau^k + (\partial_k L)\delta\tau^k + \frac{\partial L}{\partial \phi^\alpha}\delta\phi^\alpha + \frac{\partial L}{\partial y_i^\alpha}\delta y_i^\alpha \right\} \mathrm{d}V.$$

Introduction of the components of the generalized body force, f_α, and the components of the generalized stress, $\sigma_\alpha{}^i$, by (2.3) and (2.4) then results in

$$\delta A = \int_B \{LD_k\delta\tau^k + (\partial_k L)\delta\tau^k + f_\alpha\delta\phi^\alpha - \sigma_\alpha{}^i\delta y_i{}^\alpha\}\,\mathrm{d}V. \tag{2.15}$$

Finally, use of (2.12) to write δy_i^α in terms of $\{\delta\tau^k, \delta\phi^\alpha\}$ and their derivatives, together with appropriate rearrangements of the terms and the divergence theorem, lead to the evaluation

$$\delta A = \int_B \{f_\alpha + D_i\sigma_\alpha{}^i\}\,\overset{*}{\delta\phi^\alpha}\,\mathrm{d}V + \int_{\partial B} \{H_j^i\delta\tau^j - \sigma_\alpha{}^i\delta\phi^\alpha\}n_i\,\mathrm{d}S \tag{2.16}$$

where $\overset{*}{\delta\phi^\alpha} = \delta\phi^\alpha - y_i^\alpha\delta\tau^i$ and

$$-H_j^i = \frac{\partial L}{\partial y_i^\alpha}y_j^\alpha - \delta_j^i L = -\sigma_\alpha{}^i y_j^\alpha - \delta_j^i L \tag{2.17}$$

is the negative of the "momentum-energy complex" for the material body. When we recall that $L = -W - \mathcal{V}$ for elastostatics, (2.17) gives the familiar form

$$H_j^i = \sigma_\alpha{}^i y_j^\alpha - (W + \mathcal{V})\delta_j^i. \tag{2.18}$$

We have purposefully not cluttered the above discussion with the details of the calculations. The reader with an overriding interest in such matters is referred to ([1], Chap. 4; [2], Chap. 4; [3], Chap. 7; [4], Chap. 5).

3. FIELD EQUATIONS AND NATURAL BOUNDARY CONDITIONS

The classic problem of the calculus of variations deals with the requirement that the action functional be rendered *stationary* (not maximal or minimal) when there are no varaiations of the independent variables. With $\delta\tau^i = 0$, (2.16) yields the evaluation

$$\delta A = \int_B \{f_\alpha + D_i\sigma_\alpha^i\}\,\delta\phi^\alpha\,\mathrm{d}V - \int_{\partial B} \sigma_\alpha{}^i\delta\phi^\alpha n_i\,\mathrm{d}S \tag{3.1}$$

for the functional derivative of the action. Now, in order that $A[\phi]$ be rendered stationary for all variations $\{\delta\phi^\alpha\}$, it is certainly necessary that δA vanish for all $\{\delta\phi^\alpha\}$ that vanish on the boundary. Thus, we must have

$$0 = \int_B \{f_\alpha + D_i\sigma_\alpha{}^i\}\,\delta\phi^\alpha\,\mathrm{d}V \tag{3.2}$$

for all choices of the functions $\{\delta\phi^\alpha\}$ that vanish on the boundary. The fundamental lemma of the calculus of variations then gives the field equations

$$f_\alpha + D_i\sigma_\alpha{}^i = 0 \tag{3.3}$$

that must be satisfied at all points in the interior of the body. The field eqns (3.3) are just the equilibrium equations of elastostatics in the presence of body forces $\{f_\alpha\}$ and stresses $\{\sigma_\alpha{}^i\}$. They are also the Euler–Lagrange equations

$$D_k\left(\frac{\partial L}{\partial y_k^\alpha}\right) = \frac{\partial L}{\partial\phi^\alpha} \tag{3.4}$$

associated with the action functional $A[\phi^\alpha]$ with Lagrangian L.

The variational principle $\delta A = 0$ also leads directly to natural boundary conditions. When (3.3) is substituted into (3.1), we obtain

$$\delta A = \int_{\partial B} -\sigma_\alpha{}^i \delta\phi^\alpha n_i \, \mathrm{d}S. \tag{3.5}$$

Thus, the action is rendered stationary ($\delta A = 0$) under satisfaction of the field equations only when boundary conditions are imposed so as to secure satisfaction of the conditions that

$$(\sigma_\alpha{}^i \delta\phi^\alpha)|_{\partial B} n_i = 0 \tag{3.6}$$

holds at all points on ∂B. There are clearly two ways to achieve this. The first is through imposition of geometric state boundary conditions. In this instance, $\{\bar{\phi}^\alpha(\tau^m)\}$ are required to agree with $\{\phi^\alpha(\tau^m)\}$ on ∂B for all values of the parameter λ (i.e. all varied field variables assume the same prescribed boundary values). Equations (2.9) and (2.10) then show that this may be achieved if and only if $\delta\phi^\alpha$ vanish on the boundary, in which case (3.6) is also satisfied.

The second way consists of the demand that

$$\sigma_\alpha{}^i|_{\partial B} n_i = 0; \tag{3.7}$$

i.e. satisfaction of "traction free" natural boundary conditions. Under these conditions (3.6) is also satisfied. Further (3.6) can be satisfied if the boundary of the body consists of two disjoint parts; geometric state boundary conditions are specified on one part while on the other traction free natural boundary conditions are imposed.

The reader may be dismayed in finding that only "traction free" boundary conditions can be applied to the generalized stresses. Nonzero traction boundary conditions will also be obtained as natural boundary conditions, but this must wait until after the discussion of the "null class" Lagrangian density functions in Section 5.

4. TRANSVERSALITY CONDITIONS FOR PROBLEMS WITH MOVING BOUNDARIES

We now turn to the problem of the calculus of variations in the large, namely, where part or all of the boundary of the region B is allowed to change during the variation process. This situation is described by allowing variations in the independent as well as in the dependent variables, for.(2.9) and (2.10) give

$$\bar{\tau}^i = \tau^i + \lambda\delta\tau^i + 0(\lambda^2),$$

with $\delta\tau^i = v^i(\tau^m, \phi^\alpha(\tau^m))$, and the functions $v^i(\tau^m, \phi^\alpha(\tau^m))$ may be given values on ∂B that describe the departure of $\partial\bar{B}$ from ∂B to first order terms in the parameter λ.

If the action is to be rendered stationary under these circumstances (i.e. $\delta A = 0$), (2.16) yields the conditions

$$0 = \int_B \{f_\alpha + D_i\sigma_\alpha{}^i\} \overset{*}{\delta\phi}{}^\alpha \, \mathrm{d}V + \int_{\partial B} \{H_j{}^i \delta\tau^j - \sigma_\alpha{}^i \delta\phi^\alpha\} n_i \, \mathrm{d}S \tag{4.1}$$

for all variations $(\delta\tau^i, \delta\phi^\alpha)$. Since a subset of all variations consists of those for which $\{\delta\tau^i, \delta\phi^\alpha\}$ vanish on ∂B, a necessary condition is that

$$0 = \int_B \{f_\alpha + D_i\sigma_\alpha{}^i\} \overset{*}{\delta\phi}{}^\alpha \, \mathrm{d}V$$

for all $\{\overset{*}{\delta\phi}{}^\alpha\}$ that vanish on ∂B. The fundamental lemma of the calculus of variations then gives the requirements

$$0 = f_\alpha + D_i\sigma_\alpha{}^i \tag{4.2}$$

at all interior points of B. Under satisfaction of the field eqns (4.2), the requirement (4.1) reduces to

$$0 = \int_{\partial B} \{H_j{}^i \delta\tau^j - \sigma_\alpha{}^i \delta\phi^\alpha\}\, n_i \,\mathrm{d}S. \tag{4.3}$$

A set of variations $\{\delta\tau^i, \delta\phi^\alpha\}$ is said to satisfy the conditions of *transversality* on ∂B if and only if

$$H_j{}^i \delta\tau^j n_i = \sigma_\alpha{}^i \delta\phi^\alpha n_i \tag{4.4}$$

at all points of ∂B, in which case the condition (4.3) is then satisfied.

It is of particular importance to note that the transversality conditions (4.4), must be satisfied if the action is to be rendered stationary; there is no choice in the matter. In effect, what (4.4) says is that the variations in the field variables, $(\delta\phi^\alpha)$ may take only those values that are determined by the geometric variations $(\delta\tau^i)$ of the boundary, as determined through satisfaction of (4.4). Clearly, the transversality conditions are what replace the natural boundary conditions $\sigma_\alpha{}^i n_i = 0$ in the case of moving boundaries.

A further understanding may be achieved by integration of both sides of (4.4) over the boundary:

$$\int_{\partial B} H_j{}^i \delta\tau^j n_i \,\mathrm{d}S = \int_{\partial B} \sigma_\alpha{}^i \delta\phi^\alpha n_i \,\mathrm{d}S. \tag{4.5}$$

This may be read as follows: *satisfaction of the necessary transversality conditions yields the equality of the virtual "surface work" of the field variables, $\int_{\partial B} \sigma_\alpha{}^i \delta\phi^\alpha n_i \,\mathrm{d}S$, with the virtual "surface work" of the boundary displacements, $\int_{\partial B} H_j{}^i \delta\tau^j n_i \,\mathrm{d}S$.* This in turn leads to an equality between generalized field forces on variable boundaries with the actual forces that act directly in the boundaries in their real motions.

These considerations assume particular importance when the elastic body possesses one or more internal boundaries Σ. Such is the actual case when there are cracks within the material. In the case of dislocations and disclinations, such internal boundaries are introduced in the modeling process to exclude the regions occupied by the dislocations and disclinations whose presence precludes the validity of elastostatics for such regions. Again, the transversality conditions must be satisfied for such internal boundaries if the action functional is to be rendered stationary in value, in which case (4.5) gives

$$\int_\Sigma H_j{}^i \delta\tau^j N_i \,\mathrm{d}\Sigma = \int_\Sigma \sigma_\alpha{}^i \delta\phi^\alpha N_i \,\mathrm{d}\Sigma, \tag{4.6}$$

where $\{N_i\}$ are the components of the unit normal vector field to the internal boundary Σ that is oriented out of the body. Here, of course, the standard boundary conditions of elastostatics are assumed to hold on the external boundaries of the body. It is now elementary to compare (4.6) with the elegant results of Eshelby[5-7] to obtain direct analogies and identifications. The interesting thing that results here is that Eshelby's careful physical arguments (see also [8, 9]) can be reproduced directly from a stationary energy principle with due care to account for the necessary transversality conditions at internal boundaries that enclose singularities, discontinuities, or regions where the elastostatic field equations cease to hold. In fact, it is a source of wonder to this author that so much attention is paid to variational principles in elastostatics yet implementation or even recognition of the necessary transversality conditions for mobile boundaries is overlooked.

It is often claimed that relations similar to (4.5) obtain from arguments of invariance (i.e. where $\{\delta\tau^j, \delta\phi^\alpha\}$ generate an invariance transformation of the Lagrangian and hence yield a corresponding conservation law (see Section 6). Nothing could be further from the truth. Invariance transformations have to do with interior points of the body where the field equations are satisfied, while transversality is restricted solely to the boundaries of the body. Further, if $\{\delta\tau^j, \delta\phi^\alpha\}$ satisfy the necessary transversality conditions (4.4), on ∂B, then $H_j^i\delta\tau^j - \sigma_\alpha{}^i\delta\phi^\alpha$ vanish identically on ∂B. Thus, if $\{\delta\tau^j\, \delta\phi^\alpha\}$ were to be the boundary values of generators of an invariance transformation, the associated conserved current,

$$J^i = H_j^i\delta\tau^j - \sigma_\alpha{}^i\delta\phi^\alpha + Q^i,$$

(see (6.7)) would reduce to $J^i = Q^i$ on ∂B and hence arise solely from a Lagrangian density of the null class (see Section 5). The conditions of transversality and of invariance are thus essentially distinct; relations (4.4) can not be obtained from an invariance argument even though their consequences (4.5), may appear to so obtain.

5. NULL LAGRANGIANS—A QUESTION OF FACT VS FOLKLORE

We now come to an aspect of the calculus of variations that assumes the character of pure folklore in its use in elastostatics. It is a common belief that the *only* terms that can be added to a Lagrangian without a resulting change in the field equations are divergences of the form $D_kQ^k(\tau^m, \phi^\alpha(\tau^m))$. It is true that divergences of the form $D_kQ^k(\tau^m, \phi^\alpha(\tau^m))$ can be added to a Lagrangian function and there is not change in the field equations. The trouble here is that the "only" in the above statement is altogether wrong.

If two Lagrangian functions, L_1 and L_2 result in the same Euler–Lagrange equations for the determination of the ϕ^α's, then their difference, $L_1 - L_2$ is a Lagrangian function for which the Euler–Lagrange equations are satisfied identically. Lagrangians for which the Euler–Lagrange equations are satisfied identically are usually referred to as the *null class* of Lagrangian functions. Clearly, if Q is an element of the null class then

$$L_1 = L_2 + Q \tag{5.1}$$

lead to the same Euler–Lagrange equations. It is thus required that we characterize the null class of Lagrangian functions; namely, those Lagrangian functions $Q(\tau^m, \phi^\alpha(\tau^m), y_i^\alpha(\tau^m))$ for which

$$\partial Q/\partial\phi^\alpha = D_k(\partial Q/\partial y_k{}^\alpha) \tag{5.2}$$

are satisfied at every point (τ^m) of B for every choice of the functions $\phi^\alpha(\tau^m)$ that have continuous second derivatives.

The characterization of the null class goes back to the works of Carathéodory[10] (the theory of equivalent integrals) and before. A number of partial characterizations have been given at various times in the literature, but as far as the author can tell, the complete but conceptually distinct characterizations were first given in [11, 12]. The reader is also referred to the treatments given in ([1], pp. 250–261; [4], pp. 180–185).

The essence of the proof is to show that any element of the null class is a divergence, but one of the form

$$Q = D_kQ^k(\tau^m, \phi^\alpha(\tau^m), y_i^\alpha(\tau^m)) \tag{5.3}$$

with allowance for explitit dependence on the derivative variables $\{y_i^\alpha\}$. This is indeed reasonable, for Q given by (5.3) and the divergence theorem show that (5.1) yields

$$\int_B L_1\, \mathrm{d}V = \int_B L_2\, \mathrm{d}V + \int_{\partial B} Q^k n_k\, \mathrm{d}S$$

and a boundary integral does not affect the equations of motion, just the boundary conditions. The proof of this part of the result is the hard part. It will thus just be assumed, the interested reader being referred to [1, 12].

The remaining part of the characterization concerns the nature of the dependence on the derivative arguments y_i^α. Let us first note that a Lagrangian function for problems of mechanics must be independent of second derivatives. Since Q is to be a Lagrangian for mechanics problems, it must likewise be independent of second derivatives. Now, use of (1.2) to expand the total partial derivative on the r.h.s. of (5.3) gives

$$Q = \partial_k Q^k + y_k{}^\alpha \frac{\partial Q^k}{\partial \phi^\alpha} + y_{ik}^\alpha \frac{\partial Q^k}{\partial y_i^\alpha} \tag{5.4}$$

so that the only terms involving second derivatives are

$$y_{ik}^\alpha \frac{\partial Q^k}{\partial y_i^\alpha}.$$

However, $y_{ik}^\alpha \equiv y_{ki}^\alpha$ (see (1.3)), and hence the terms involving second derivatives will vanish identically whenever the coefficients of y_{ik}^α are antisymmetric in i and k; that is

$$\partial Q^k/\partial y_i^\alpha = -\partial Q^i/\partial y_k{}^\alpha. \tag{5.5}$$

These are the conditions that govern the dependence of the Q^k's derivatives. Their general solution is quite easy to obtain—the Q^k's can depend on the y_i^α only through polynomials in these variables whose coefficients are completely skewsymmetric in all lower case Latin indices and hence their degree is at most one less than the dimension of the space of independent variables. To write down the solution in the general case is an unnecessary bother since the dimension of the space of independent variables is three for elastostatic problems. For $n = 3$, we have

$$Q^k = S^k(\tau, \phi) + S_\alpha{}^{ki}(\tau, \phi) y_i^\alpha + S_{\alpha\beta}^{kij}(\tau, \phi) y_i^\alpha y_j^\beta$$

where $S_\alpha{}^{ki}(\tau, \phi)$ are antisymmetric in the indices (k, i) and $S_{\alpha\beta}^{kij}(\tau, \phi)$ are completely antisymmetric in the indices (k, i, j). When this is substituted back into (5.3), we see that *the most general Lagrangian of the null class for elastostatic problems is given by*

$$Q = \partial_k S^k + \left(\frac{\partial S^k}{\partial \phi^\gamma} + \partial_m S_\gamma{}^{mk}\right) y_k{}^\gamma + \left(\frac{\partial S_\alpha{}^{ki}}{\partial \phi^\gamma} + \partial_m S_{\alpha\gamma}^{mik}\right) y_k{}^\gamma y_i^\alpha + \frac{\partial S_{\alpha\beta}^{kij}}{\partial \phi^\gamma} y_k{}^\gamma y_i^\alpha y_j^\beta. \tag{5.6}$$

With $S_\alpha{}^{ki} = 0$, $S_{\alpha\beta}^{kij} = 0$, (5.6) reduces to the case $Q = D_k S^k(\tau^m, \phi^\alpha(\tau^m))$ that folklore would have us believe to be all that can ever occur.

The first, and simplest application of these results is to use them to replace the traction free natural boundary conditions by given traction boundary conditions that are also natural. If we replace the Lagrangian L of the problem by $L + Q$, for Q given by (5.6), then the field equations,

$$0 = f_\alpha + D_k \sigma_\alpha{}^k, f_\alpha = \partial L/\partial \phi^\alpha, \sigma_\alpha{}^k = -\partial L/\partial y_i^\alpha$$

go over into

$$0 = f_\alpha + D_k \sigma_\alpha{}^k + \dot{f}_\alpha + D_k \dot{\sigma}_\alpha{}^k$$

with f_α and $\sigma_\alpha{}^k$ as before and

$$\dot{f}_\alpha = \partial Q/\partial \phi^\alpha, \dot{\sigma}_\alpha{}^k = -\partial Q/\partial y_i^\alpha.$$

However, $\overset{*}{f}_\alpha + \overset{*}{D_k\sigma_\alpha}{}^k$ vanishes identically for any and all C^2 choices of the field variables $\phi^\alpha(\tau^m)$ and hence the field equations assume their previous form

$$0 = f_\alpha + D_k\sigma_\alpha{}^k.$$

The situation is quite different for the boundary integral, for $\overset{*}{\sigma}_\alpha{}^k$ does not vanish identically so we obtain the *new* condition

$$\int_{\partial B} (\overset{*}{\sigma}_\alpha{}^i + \sigma_\alpha{}^i) n_i \delta\phi^\alpha \, dS = 0.$$

This leads to the new "surface traction" boundary conditions

$$\sigma_\alpha{}^i|_{\partial B} n_i = -\overset{*}{\sigma}_\alpha{}^i|_{\partial B} n_i \overset{\text{def}}{=} T_\alpha|_{\partial B} \tag{5.7}$$

where the quantities $T_\alpha(\tau^m, \phi^\alpha(\tau^m), y_i^\alpha(\tau^m))$ may be specified on ∂B. For the choice $S^k = 0$, $S_{\alpha\beta}^{kij} = 0$, $S_\gamma{}^{mk} = g_\gamma{}^{mk}(\tau^i)$, Q reduces to $Q = y_k{}^\gamma \partial_m g_\gamma{}^{mk}(\tau^i)$ and we have $\overset{*}{\sigma}_\gamma{}^k = \partial_m g_\gamma{}^{mk}(\tau^i)$, $g_\gamma{}^{mk}(\tau) = -g_\gamma{}^{km}(\tau)$. In this instance, (5.7) reduces to

$$\sigma_\alpha{}^i{}_{\partial B} n_i = -n_i \partial_m g_\alpha{}^{mi}(\tau^j)|_{\partial B} = T_\alpha(\tau^j)|_{\partial B}.$$

However, $D_i\overset{*}{\sigma}_\alpha{}^i = \partial_i\partial_m g_\alpha{}^{mi}(\tau^j) \equiv 0$, and hence any choice of the functions $g_\alpha{}^{mk}(\tau^i)$ that satisfy

$$n_i \partial_m g_\alpha{}^{mi}(\tau^j)|_{\partial B} = T_\alpha(\tau^j)|_{\partial B},\ g_\alpha{}^{mk}(\tau^j) = -g_\alpha{}^{km}(\tau^j)$$

will lead to the natural traction boundary conditions

$$\sigma_\alpha{}^i|_{\partial B} n_i = T_\alpha(\tau^m)|_{\partial B}.$$

It should also be remarked that (5.6) allows situations in which the tractions applied to the boundary depend on the boundary values of the ϕ^α's and on the y_i^α's (i.e. where the boundary is constrained by translational and rotational springs in addition to applied tractions).

If one thinks about these problems, the above results become physically as well as mathematically plausible. This follows from the fact that $Q = D_k Q^k$ and hence the added action, $\int_B Q \, dV$, can equally well be evaluated in terms of the surface integral $\int_{\partial B} Q^k n_k \, dS$. Thus the replacement of L by $L + Q$ is the same thing as adding a surface integral that accounts for the work done on the body by the applied surface tractions.

6. INVARIANCE OF THE ACTION AND CONSERVATION LAWS

We now come to another area in which folklore seems to hold sway, namely, in considerations of invariance of the action functional under general transformations of the type given by (2.5) and the associated conservation laws admitted by solutions of the field equations. Again, it is not a question of errors in the calculations, rather, the claims that are made about the results. Clearly, when one makes the assertion that there are exactly 7 or so conservation laws of elastostatics, the results had better obtain from a fully general setting of the problem rather than from only a special case.

If we subject the independent and dependent variables to a specific family of transformations of the form given by (2.5).

$$\bar{\tau}^i(\lambda) = P^i(\tau^m, \phi^\beta; \lambda),\ \bar{\phi}^\alpha(\lambda) = P^\alpha(\tau^m, \phi^\beta; \lambda), \tag{6.1}$$

it may happen that the action functional remains invariant in value for each choice of the field variables and of the domain, B, of the body. In this event everyone seems to be aware that there is then as associated conservation law that is satisfied by any solution of the field equations and the natural boundary conditions. We saw in the last section, however, that changes in the system through addition of a surface integral of the appropriate type leaves the field equations unchanged. Thus, if the evaluation of the action functional in the state $\bar{\phi}^\alpha$ results in a change that comes about only through a surface integral that also leaves the boundary conditions for the problem unchanged, then we must likewise expect the emergence of an associated conservation law. Now, the invariance of the boundary conditions requires that the added surface integral vanish at $\lambda = 0$, and hence we need to consider the case where

$$A[\bar{\phi}^\alpha] = A[\phi^\alpha] + \lambda \int_B D_k Q^k \,\mathrm{d}V \tag{6.2}$$

and $D_k Q^k$ is an element of the null class of Lagrangian functions.

It must be carefully realized that we have now turned the problem around. Before, we allowed the functions P^i and P^α in (6.1) to be arbitrary so as to generate a sufficiently rich family of variations. What we must now do is to allow the ϕ^α's to be arbitrary and to seek conditions on the functions P^i and P^α so as to secure satisfaction of (6.2). Clearly, if the transformations (6.1) are to satisfy the condition (6.2), then they must satisfy it to first order terms in the parameter λ. The beautiful thing here is that satisfaction of the condition (6.2) to first order in λ implies that the functions P^i and P^α are then solutions of the eqns (2.7) where the functions v^i and v^α appear as

$$\bar{\tau}^i = \tau^i + \lambda v^i(\tau^m, \phi^\beta) + O(\lambda^2),$$
$$\bar{\Phi}^\alpha = \Phi^\alpha + \lambda v^\alpha(\tau^m, \phi^\beta) + O(\lambda^2),$$

and hence are finite realizations of the action of one parameter groups. A little further effort then shows that all solutions to this problem obtain through realizations of the transformations (6.1) as 1-parameter orbits of a Lie group. This means that satisfaction of the conditions (6.2) to first order terms in λ implies satisfaction to all orders in λ (see [4], Theorem 5.15, p. 149).

We now simply substitute (2.14), to first order terms in λ into (6.2). This gives the requirement

$$0 = \int_B \left\{ L D_k \delta\tau^k + (\partial_k L)\delta\tau^k + \frac{\partial L}{\partial \phi^\alpha}\delta\phi^\alpha + \frac{\partial L}{\partial y_i^\alpha}\delta y_i^\alpha + D_k Q^k \right\} \mathrm{d}V. \tag{6.3}$$

If this condition is to hold for the whole elastic body B, then it holds as a consequence of the material structure of the body, in which case it should also hold for every subbody b of B. This additional requirement, that (6.3) continue to hold when B is replaced by any subbody b, is classic and obtains in all local field theories from the very foundations of the subject in the now famous paper by Noether[13]. Such an argument is certainly necessary in order to conclude that the integrand that appears in (6.3) should vanish at each point of B. Equating the integrand in (6.3) to zero and use of (2.12) to evaluate δy_i^α then leads to the following equations for the determination of the required quantities $(\delta\tau^i, \delta\phi^\alpha)$:

$$L D_k \delta\tau^k + (\partial_k L)\delta\tau^k + \frac{\partial L}{\partial \phi^\alpha}\delta\phi^\alpha + \frac{\partial L}{\partial y_i^\alpha}\left(\partial_i + y_i^\beta \frac{\partial}{\partial \phi^\beta}\right)(\delta\phi^\alpha - y_k^\alpha \delta\tau^k) + D_k Q^k = 0. \tag{6.4}$$

Written in terms of $f_\alpha = \partial L/\partial\phi^\alpha$ and $\sigma_\alpha{}^i = -\partial L/\partial y_i^\alpha$, they become

$$L D_k \delta\tau^k + (\partial_k L)\delta\tau^k + f_\alpha \delta\phi^\alpha - \sigma_\alpha{}^i\left(\partial_i + y_i^\beta \frac{\partial}{\partial \phi^\beta}\right)(\delta\phi^\alpha - y_k^\alpha \delta\tau^k) + D_k Q^k = 0. \tag{6.5}$$

It is essential to note that the quantities $(\delta\tau^i, \delta\phi^\alpha)$ must be functions only of the arguments (τ^m, ϕ^β), and hence all coefficients of all powers of the y's that occur in (6.5) must vanish separately. It is this identical satisfaction of (6.5) in the variables y_i^α that leads to the numerous equations that finally serve to determine the possible forms of the functions $(\delta\tau^i, \delta\phi^\alpha)$.

It might happen that the only solution is $\delta\tau^i = 0$, $\delta\phi^\alpha = 0$, in which case nothing more can be said. As it turns out, most real problems admit nontrivial solutions of (6.5), in which case we can ask what implications may then be drawn. Let us suppose that we know a solution $(\delta\tau^i, \delta\phi^\alpha)$. A simple rearrangement of the various terms in (6.5), similar to that used in obtaining (2.16) from (2.14), leads to the relations

$$0 = \{f_\alpha + D_i\sigma_\alpha{}^i\} \overset{\bullet}{\delta}\phi^\alpha + D_i \{H_j{}^i\delta\tau^j - \sigma_\alpha{}^i\delta\phi^\alpha + Q^i\} \tag{6.6}$$

Thus, *any solution of the field equations* $f_\alpha + D_i\sigma_\alpha{}^i = 0$ *will give identical satisfaction of the conservation law*

$$D_i \{H_j{}^i\delta\tau^j - \sigma_\alpha{}^i\delta\phi^\alpha + Q^i\} = 0. \tag{6.7}$$

In point of fact, there will be as many independent conservation laws as there are linearly independent solutions of the invariance conditions (6.5).

It is important to note in this context that each solution of the invariance condition (6.5) will have its own associated Q^i. The situation that seems to obtain in most of the literature in elastostatics is where the Q^i have been set to zero identically. This is clearly a severe restriction and precludes all those solutions and associated conservation laws with nonzero Q^i. A claim based upon $Q^i = 0$ of the total number of conservation laws is thus clearly false, although each conservation law so obtained is indeed one. The reader is invited to experiment here, for even in the case of a quadratic Lagrangian function (linear elasticity), there are indeed nontrivial invariance transformations and conservation laws associated with nonvanishing Q^i's. In fact, a detailed cataloging of *all* invariance transformations and conservation laws in linear elasticity would seem a worthy task.

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